MATHEMATICS

DUALITY AND HARMONIC ANALYSIS ON CENTRAL TOPOLOGICAL GROUPS. I

BY

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In this paper we continue the study ([8], [9], [10]) of central topological groups G. In § 1 we study the local properties of the dual, $G^\wedge$. We prove (1.4) that the canonical map λ: $G^\wedge \to Z^\wedge$ is a local homeomorphism. An inductive argument, which uses the fact (1.2) that certain "constituents" of λ are in fact covering projections, plays a major role in the proof of this as well as many other results throughout the paper. In § 2 we prove a duality theorem (2.2) which characterizes topological group theoretic properties of G in terms of topological properties of $G^\wedge$. This theorem generalizes the classical results of Pontrjagin and others on abelian groups. § 3 contains further results on harmonic analysis. We prove a generalized Fourier inversion theorem (3.3) and Poisson summation formula (3.4). We also give a generalization to $[Z]$-groups of the Wiener Tauberian theorem (3.8). In § 4 we construct and study examples of $[Z]$-groups which illustrate some of the results in the paper.

The notation in the paper generally agrees with that of [8], with the following notable exception: we now use $G^\wedge$ to denote the set of equivalence classes of continuous irreducible unitary representations of G. As usual, of course, G always denotes a locally compact group, and Z (or $Z(G)$) its center. If f is any mapping defined on G, and S is a subset of G, we denote the restriction of f to S by f_S.

§ 1. Local properties of the dual

In the present study of $[Z]$-groups an inductive argument very often plays a significant role, and the inductive step can in some instances be isolated so as to clarify this role. We shall therefore begin our discussion with the inductive step, that is, with the case of a $[Z]$-group G containing a normal subgroup H of prime index p. The entire discussion through (1.2) will be assumed to deal with this situation.

For the most elementary facts we need assume at first only that G is a locally compact group whose continuous irreducible unitary representations are all finite-dimensional. We shall show that $G^\wedge$ and $H^\wedge$ divide naturally into corresponding disjoint subsets, consisting in the case of

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$G^\wedge$ of (I): irreducible representations of G which restrict irreducibly to H; and (II): irreducible representations of G which are induced by irreducible representations of H. The basis for our analysis is of course Clifford's theorem [1]) [3, (49.2)], which asserts that if $\varrho \in G^\wedge$ then the restriction ϱ_H is a multiple of the direct sum $\bigoplus \sigma^x$ (taken over the distinct conjugates of σ), where σ is a fixed (but arbitrary) element of $H^\wedge$ contained in ϱ_H. Since the stability group $S(\sigma)$ of any $\sigma \in H^\wedge$ must satisfy $G \supseteq S(\sigma) \supseteq H$, and since $[G: H]$ is prime, it follows that either (I)': $S(\sigma)=G$, and ϱ_H is simply a multiple of σ; or (II)': $S(\sigma)=H$, and ϱ_H is a multiple of $\bigoplus \sigma^x$ $(\dot{x} \in G/H)$.

Suppose first that σ is any element of $H^\wedge$ such that $S(\sigma)=G$. Then σ extends to a continuous cocycle representation of G (see [17, Theorem 8.2]) obtained from an element of $H^2(Z_p, \mathbf{T})$. Now any central cyclic extension of $\mathbf{T}$ must be abelian and therefore split (since $\mathbf{T}$ is injective in the category of locally compact abelian groups), so $H^2(Z_p, \mathbf{T})=0$. Thus σ actually extends to an ordinary (irreducible) unitary representation τ of G. Moreover, if $\chi_1^\sim, \ldots, \chi_p^\sim$ are the p distinct linear characters of G/H lifted to G, then $\{\chi_1^\sim \otimes \tau, \ldots, \chi_p^\sim \otimes \tau\}$ is the full set of extensions (or, more properly, of equivalence classes of extensions) of σ. In fact, we can observe first that these representations are mutually inequivalent. Since $(G/H)^\wedge$ is a group, to establish this assertion it clearly suffices to show that $\chi \in (G/H)^\wedge$ and $\chi^\sim \otimes \tau \simeq \tau$ imply $\chi^\sim \equiv 1$. Let χ_τ denote the character of τ; then $\chi^\sim(x)\ \chi_\tau(x)=\chi_\tau(x)$, so $\chi_\tau(x)=0$ whenever $\chi^\sim(x) \neq 1$. Now $\chi^\sim(x)$ cannot equal 1, for any $x \notin H$, unless $\chi^\sim \equiv 1$; for otherwise $\chi^\sim$ takes a generator of G/H onto 1, and consequently $\chi^\sim$ is identically 1. Thus we may assume that $\chi^\sim(x) \neq 1$ for each $x \notin H$, so χ_τ vanishes off H. In this case τ must in fact be the induced representation σ^G; for the character of σ^G is given by

$$\chi_{\sigma^G}(t)=[G: H]^{-1}\,\xi_H(t) \sum_{\dot{x} \in G/H} \chi_\sigma(xtx^{-1})$$

(where ξ_H denotes the characteristic function of H), and the invariance of σ therefore implies that $\chi_{\sigma^G}=\xi_H\chi_\sigma=\xi_H\chi_\tau=\chi_\tau$. But this contradicts the fact that $\sigma=\tau_H$ is irreducible, and we therefore conclude that $\chi^\sim \equiv 1$. Thus the p representations $\chi_i^\sim \otimes \tau$ are mutually inequivalent; clearly each one restricts to σ, and therefore by the Frobenius reciprocity theorem [3, (38.8)] σ^G must contain $\bigoplus_{i=1}^p \chi_i^\sim \otimes \tau$. Comparison of degrees shows that this containment is actually equality. Therefore, again by the Frobenius reciprocity theorem, any $\varrho \in G^\wedge$ which restricts to σ, or which even has σ as an irreducible component of its restriction to H, must be of the form $\chi^\sim \otimes \tau$, for some $\chi \in (G/H)^\wedge$. In particular, if $\varrho \in G^\wedge$ and ϱ_H

[1]) The results concerning Clifford's theorem and the Frobenius Reciprocity Theorem are most often stated in the literature either for finite groups and finite-dimensional representations or for separable locally compact groups. Nevertheless, in the context with which we are concerned here and below, that is, with finite-dimensional representations and subgroups of finite index, all the theorems are true just as in the case of finite groups.

is a multiple of some $\sigma \in H^\wedge$, then the above analysis shows that in fact $\varrho_H = \sigma$. Thus the multiple in condition (I)′ is exactly 1.

Consider, on the other hand, a $\sigma \in H^\wedge$ such that $S(\sigma) = H$; then σ^G is irreducible. If σ is a component of the restriction ϱ_H of some $\varrho \in G^\wedge$, then ϱ is contained in σ^G and therefore equals σ^G. It follows that the multiple in condition (II)′ is also exactly 1. Let us define $(G^\wedge)_I = \{\varrho \in G^\wedge : \varrho_H \text{ is irreducible}\}$, $(G^\wedge)_{II} = \{\varrho \in G^\wedge : \varrho = \sigma^G \text{ for some } \sigma \in H^\wedge\}$. Also, let $(H^\wedge)_I = \{\sigma \in H^\wedge : S(\sigma) = G\}$, $(H^\wedge)_{II} = \{\sigma \in H^\wedge : S(\sigma) = H\}$. Finally, let $\varkappa_* : (G^\wedge)_I \to (H^\wedge)_I$ be the restriction $\varrho \mapsto \varrho_H$. The above discussion can then be summed up, in a slightly different formulation, in the following lemma.

(1.1) LEMMA: $G^\wedge$ is the disjoint union of $(G^\wedge)_I$ and $(G^\wedge)_{II}$; $H^\wedge$ is the disjoint union of $(H^\wedge)_I$ and $(H^\wedge)_{II}$. Moreover, we have:

(I): If $\sigma \in H^\wedge$, then $\sigma \in (H^\wedge)_I$ if and only if $\sigma = \varrho_H$ for some $\varrho \in (G^\wedge)_I$. In this case all the extensions of σ are of the form $\chi^\sim \otimes \varrho$, for $\chi \in (G/H)^\wedge$. The action of $(G/H)^\wedge$ on $(G^\wedge)_I$ via $(\chi, \varrho) \mapsto \chi^\sim \otimes \varrho$ is fixed point free, and the surjective map $\varkappa_* : (G^\wedge)_I \to (H^\wedge)_I$ induces a bijection of the orbit space $(G^\wedge)_I/(G/H)^\wedge$ onto $(H^\wedge)_I$.

(II): If $\sigma \in H^\wedge$, then $\sigma \in (H^\wedge)_{II}$ if and only if σ^G is irreducible. Moreover, for $\varrho \in G^\wedge$, $\varrho \in (G^\wedge)_{II}$ if and only if $\varrho_H = \bigoplus \sigma^x$ $(\dot{x} \in G/H)$, and this is the case if and only if $\chi_\varrho(t) \equiv 0$ for $t \in G - H$.

Suppose now in addition that G is a [Z]-group. In this case $G^\wedge$ can be equipped with a naturally defined locally compact Hausdorff topology, described as follows (see [8, Theorem (6.6)]). $G^\wedge$ is in bijective correspondence with the set of normalized characters $\mathfrak{X}(G) = \{d_\varrho^{-1} \chi_\varrho : \varrho \in G^\wedge\}$, and therefore the topology on the latter, defined by uniform convergence on compacta of G, can be transported to the former. Moreover, since the degree function $\varrho \to d_\varrho$ of $G^\wedge \to \mathbb{Z}_+$ is continuous, this topology coincides with the topology of uniform convergence on compacta for the unnormalized characters $\{\chi_\varrho : \varrho \in G^\wedge\}$. Finally, in view of [19, (5.11) and (5.12)], this topology on $G^\wedge$ also coincides with the Fell topology.

In order to tie in the topological considerations with our decomposition of $G^\wedge$ and $H^\wedge$, we introduce, as in [19], the set $\mathfrak{X}^G(H)$ [1]). $\mathfrak{X}^G(H)$ consists of the non-zero extreme points in the set of continuous, positive definite functions φ on H which are G-invariant ($\varphi^x(h) = \varphi(h)$ for all $h \in H$, $x \in G$, where $\varphi^x(h) = \varphi(xhx^{-1})$) and satisfy $\varphi(1) \leqslant 1$. $\mathfrak{X}^G(H)$ is a locally compact Hausdorff space in the topology of uniform convergence on compacta of H [19, Corollary 4.2]. Both $G^\wedge$ and $H^\wedge$ map surjectively onto $\mathfrak{X}^G(H)$. In fact, if $\varrho \in G^\wedge$, let $\varphi_\varrho = d_\varrho^{-1} \chi_\varrho$, and let $\varkappa(\varrho) = (\varphi_\varrho)_H$ be the restriction to H. Then by [19, Proposition 2.9] $\varkappa$ is a continuous map of $G^\wedge$ *onto* $\mathfrak{X}^G(H)$.

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1) The notation here is slightly different from that in [19]. We remark that in view of [19, Theorem 5.8] one could also work with the orbit space $H^\wedge/G$, replacing $\mathfrak{X}^G(H)$. However, it would be difficult to prove directly, without using $\mathfrak{X}^G(H)$, that $\varkappa$ is a continuous map. It thus seems that although we are dealing only with [Z]-groups, the theory of [Z]-groups does not suffice to prove all the desired results.

On the other hand, for $\sigma \in H^\wedge$, let $\pi(\sigma) = p^{-1} \sum (\varphi_\sigma)^s$ ($\dot{s} \in G/H$), where $\varphi_\sigma = d_\sigma^{-1} \chi_\sigma$. Then by [19, Theorem 5.8] π is a continuous, open, closed, and proper mapping (that is, the inverse image of compact sets is compact) of $H^\wedge$ *onto* $\mathfrak{X}^G(H)$.

Now, let $\mathfrak{X}^G(H)_I = \mathfrak{X}^G(H) \cap \mathfrak{X}(H)$, and let $\mathfrak{X}^G(H)_{II} = \mathfrak{X}^G(H) - \mathfrak{X}^G(H)_I$. Then $\varkappa^{-1}(\mathfrak{X}^G(H)_I)$ is just the set of $\varrho \in (G^\wedge)_I$ such that $(\chi_\varrho)_H$ is the character of an irreducible representation of H; thus $\varkappa^{-1}(\mathfrak{X}^G(H)_I)$ is exactly $(G^\wedge)_I$. Since $\varkappa$ is surjective, it follows that $\varkappa((G^\wedge)_I) = \mathfrak{X}^G(H)_I$, and $\varkappa((G^\wedge)_{II}) = \mathfrak{X}^G(H)_{II}$. Similarly, $\pi^{-1}(\mathfrak{X}^G(H)_I)$ is just the set of $\sigma \in H^\wedge$ such that $p^{-1} \sum (\varphi_\sigma)^s$ is a normalized character of H. But normalized characters cannot be expressed as proper convex combinations of the above type, so this must imply $(\varphi_\sigma)^s = \varphi_\sigma$ for all $s \in G$. Thus $\pi^{-1}(\mathfrak{X}^G(H)_I) = (H^\wedge)_I$; again it follows that $\pi((H^\wedge)_I) = \mathfrak{X}^G(H)_I$ and $\pi((H^\wedge)_{II}) = \mathfrak{X}^G(H)_{II}$. Thus the diagram of maps

$$\begin{tikzcd} G^\wedge \arrow[dr, "\varkappa"'] & & H^\wedge \arrow[dl, "\pi"] \\ & \mathfrak{X}^G(H) & \end{tikzcd}$$

decomposes into the two commutative diagrams

$$\begin{tikzcd} (G^\wedge)_I \arrow[rr, "\varkappa_*"] \arrow[dr, "\varkappa_I"'] & & (H^\wedge)_I \arrow[dl, "\pi_I", "\cong"'] \\ & \mathfrak{X}^G(H)_I & \end{tikzcd}$$

and

$$\begin{tikzcd} (G^\wedge)_{II} \arrow[dr, "\varkappa_{II}"', "\cong"] & & (H^\wedge)_{II} \arrow[ll, "\pi_*"'] \arrow[dl, "\pi_{II}"] \\ & \mathfrak{X}^G(H)_{II} & \end{tikzcd}$$

(π_* is the map $\sigma \mapsto \sigma^G$). We note that π_I and $\varkappa_{II}$ are bijections. In fact, if $\sigma \in (H^\wedge)_I$, then $\pi(\sigma) = \varphi_\sigma$, since σ is G-invariant; thus π_I is simply the map sending a representation to its normalized character. Similarly, if $\varrho \in (G^\wedge)_{II}$, then $\chi_\varrho \equiv 0$ off H, and therefore the restriction map $\varkappa_{II}$ is injective. We can now state

(1.2) Proposition: Let G be a [Z]-group. The sets $(G^\wedge)_I$ and $(G^\wedge)_{II}$ are complementary open and closed subsets of $G^\wedge$; similarly, $(H^\wedge)_I$ and $(H^\wedge)_{II}$ are complementary open and closed subsets of $H^\wedge$. The maps $\varkappa\colon G^\wedge \to \mathfrak{X}^G(H)$, $\pi\colon H^\wedge \to \mathfrak{X}^G(H)$ are open, closed, and proper surjections; π_I and $\varkappa_{II}$ are homeomorphisms. $\varkappa_*\colon (G^\wedge)_I \to (H^\wedge)_I$ is a covering projection (in the sense of [25]) and the fiber over each point is $(G/H)^\wedge$.

PROOF: $(G^\wedge)_I$ is characterized by the condition "$(\chi_\varrho)_H$ is the character of an irreducible representation of H," and therefore, equivalently, by the condition "χ_ϱ satisfies the character formula on H" (see [8, Theorem (1.5)]). Since this last condition is clearly preserved under uniform convergence on compacta, it follows that $(G^\wedge)_I$ is closed. Similarly, it follows that $(G^\wedge)_{II}$, characterized by the condition "$\chi_\varrho \equiv 0$ off H," is closed in $G^\wedge$. This establishes the first assertion. We have already remarked that π is a continuous, open, closed, and proper surjection [19, Theorem 5.8], and similarly, that $\varkappa$ is a continuous surjection. Now the one-point compactifications of $G^\wedge$ and $\mathfrak{X}^G(H)$ can be identified with the subsets $\mathfrak{X}(G) \cup (0) \subset L_\infty(G)$ and $\mathfrak{X}^G(H) \cup (0) \subset L_\infty(H)$ respectively ([19, Corollary 4.2]). It follows immediately that $\varkappa$ extends to a continuous map "at infinity." Therefore $\varkappa$ is also a closed and proper map onto $\mathfrak{X}^G(H)$ (see [2a, ch. 1, § 10.3]). But this implies that $\mathfrak{X}^G(H)_I = \varkappa((G^\wedge)_I)$ and $\mathfrak{X}^G(H)_{II} = \varkappa((G^\wedge)_{II})$ are closed subsets of $\mathfrak{X}^G(H)$, and thus that $(H^\wedge)_I = \pi^{-1}(\mathfrak{X}^G(H)_I)$ and $(H^\wedge)_{II} = \pi^{-1}(\mathfrak{X}^G(H)_{II})$ are closed in $H^\wedge$. Moreover, $\varkappa_{II}$ and π_I are clearly homeomorphisms, since we have already remarked that they are bijections.

Now, since $\varkappa$ maps the open subset $(G^\wedge)_{II}$ homeomorphically onto an open subset of $\mathfrak{X}^G(H)$, in order to show that $\varkappa$ is an open mapping it will suffice to show that $\varkappa$ maps open subsets of $(G^\wedge)_I$ onto open subsets of $\mathfrak{X}^G(H)_I$. Since a covering projection is an open mapping, the proof of the proposition will be complete when we establish the last assertion. Now $\varkappa_*$ can be regarded as the projection of $(G^\wedge)_I$ onto its orbit space under the action of $(G/H)^\wedge$ (see Lemma (1.1)); since $\varkappa_*$ is continuous and closed, the topology on $(H^\wedge)_I$ is the quotient topology. Thus $(G/H)^\wedge$ is a finite group acting fixed point free on the Hausdorff space $(G^\wedge)_I$, with orbit space $(H^\wedge)_I$. If $\mathfrak{U}$ is an open neighborhood in $(G^\wedge)_I$ such that $(\chi^\sim \otimes \mathfrak{U}) \cap \cap ((\chi')^\sim \otimes \mathfrak{U}) \neq \phi$ implies $\chi \equiv \chi'$ $(\chi, \chi' \in (G/H)^\wedge)$, then $\varkappa_*(\mathfrak{U})$ is an open neighborhood in $(H^\wedge)_I$ which is evenly covered, and neighborhoods of this form cover $(H^\wedge)_I$ (see [25, Theorem (2.6.7)]).

For later use we record here an elementary lemma.

(1.3) LEMMA: Let G be any locally compact group of the form $G = KH$, where K is a compact subgroup and H any closed subgroup. Then every compact subset C of G is contained in a set of the form KF, where F is a compact set in H.

PROOF: Since K is compact the Second Isomorphism Theorem shows that the restriction to H of the canonical surjection $\pi\colon G \to G/K$ is open. This implies that if U is open in H then KU is open in G. Since the sets KU, as U varies over the compact neighborhoods in H, cover G, there is a finite family $\{U_i\}$ such that $C \subseteq \bigcup_i KU_i = K(\bigcup_i U_i) = KF$, where F is compact in H.

We have defined in [10, section 6], a canonical map $\lambda = \lambda_G\colon G^\wedge \to Z^\wedge$, which to each irreducible representation ϱ of G assigns the linear character of $Z = Z(G)$ obtained by restriction (here, as always in this paper, Z will denote the center of G). We know that λ is a surjection [10, Theorem 5.5] if G is a $[Z]$-group. The following theorem, which gives further properties of λ, plays an important role in the study of the dual spaces of $[Z]$-groups.

(1.4) THEOREM: Let G be a $[Z]$-group. The canonical map $\lambda\colon G^\wedge \to Z^\wedge$ is a local homeomorphism. In particular, λ is an open map.

PROOF: We first show that λ is locally injective. In fact, write $G = UZ$ as in [10], with U a compact neighborhood of 1, fix $\varrho_0 \in G^\wedge$, and let $\mathfrak{W} = \mathfrak{W}(U, \varrho_0, \frac{1}{2})$ be the neighborhood of ϱ_0 defined by: $\varrho \in \mathfrak{W}$ iff $\|\chi_\varrho - \chi_{\varrho_0}\|_U < \frac{1}{2}$. If $\varrho_1, \varrho_2 \in \mathfrak{W}$, then $\|\chi_{\varrho_1} - \chi_{\varrho_2}\|_U < 1$. Suppose that $\lambda(\varrho_1) = \chi = \lambda(\varrho_2)$; then if $x \in G$, $x = uz$ with $u \in U$ and $z \in Z$, and $\chi_{\varrho_i}(x) = \chi_{\varrho_i}(u)\chi(z)$ $(i = 1, 2)$; since $|\chi| = 1$ it follows that $\|\chi_{\varrho_1} - \chi_{\varrho_2}\|_G < 1$. Thus $\varrho_1 = \varrho_2$ (as in, [8, Theorem 5.1]) and λ is injective on $\mathfrak{W}$.

Since λ is locally injective it clearly suffices to prove that λ is open. We shall do this by using an inductive procedure which we now describe. By the structure theorem [9, Theorem 4.4 and Corollary 2], $G = V \times M$ (direct product) where $M \supseteq K$, K a compact open normal subgroup of M containing $(G')^-$. It follows that KZ is a normal subgroup of G with finite index, and with an abelian quotient $G/(KZ)$. As a finite abelian group $G/(KZ)$ has a composition series whose successive quotients are cyclic of prime order. By lifting this composition series back up to G we get a finite chain of normal subgroups of G containing KZ, and with successive quotients cyclic of prime order. The proof of the theorem will be given by induction on the length r of this chain.

We deal first with the case $r = 0$, that is, $G = KZ$ with K a compact subgroup. Let C be a compact set in G, and $\varrho, \varrho_0 \in G^\wedge$. Then $C \subseteq KF$ (as in (1.3)) and

$$
\begin{aligned}
\|\chi_\varrho - \chi_{\varrho_0}\|_C &\leqslant \|\chi_\varrho - \chi_{\varrho_0}\|_{KF} \\
&= \sup_{z \in F, k \in K} |\lambda_\varrho(z)\chi_\varrho(k) - \lambda_{\varrho_0}(z)\chi_{\varrho_0}(k)| \\
&\leqslant \sup |\lambda_\varrho(z)\chi_\varrho(k) - \lambda_\varrho(z)\chi_{\varrho_0}(k)| + \sup |\lambda_\varrho(z)\chi_{\varrho_0}(k) - \lambda_{\varrho_0}(z)\chi_{\varrho_0}(k)| \\
&\leqslant \|\lambda_\varrho\|_F \|\chi_\varrho - \chi_{\varrho_0}\|_K + \|\chi_{\varrho_0}\|_K \|\lambda_\varrho - \lambda_{\varrho_0}\|_F.
\end{aligned}
$$

Now $\|\lambda_\varrho\|_F = 1$ and $\|\chi_{\varrho_0}\|_K \leqslant \|\chi_{\varrho_0}\|_G = d_{\varrho_0}$. Hence if $\lambda_\varrho \to \lambda_{\varrho_0}$ uniformly on compacta of Z, and $\varrho \in \mathfrak{W}(K, \varrho_0, \frac{1}{2})$, then $\chi_\varrho \to \chi_{\varrho_0}$ uniformly on C (since, as in [8, Theorem 5.1], $\|\chi_\varrho - \chi_{\varrho_0}\|_K$ actually equals 0). Thus λ is a local homeomorphism in this case.

Assume now that the theorem holds for all $[Z]$ groups which have a normal chain of the type described above of length less than r, and suppose $G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_r = KZ$, with G_i/G_{i+1} cyclic of prime order $(i = 0, \ldots, r-1)$. If $H = G_1$, then H is a $[Z]$-group and $Z = Z(G) \subseteq Z(H)$, so that the inductive hypothesis applies to H. Thus the map $\lambda_H\colon H^\wedge \to Z(H)^\wedge$ is open, and since the restriction homomorphism $i^*\colon Z(H)^\wedge \to Z^\wedge$ is open (by standard duality theory for locally compact abelian groups), the composition $\lambda_H^* = i^* \circ \lambda_H\colon H^\wedge \to Z^\wedge$ is open. We recall now the space $\mathfrak{X}^G(H)$ defined in the discussion preceding Proposition (1.2). It follows from the G-character formula [19, Proposition 4.4] that the restriction to $Z(G)$ of any $\varphi \in \mathfrak{X}^G(H)$ is a linear character of $Z(G)$. If we denote the map $\varphi \mapsto \varphi_Z$ of $\mathfrak{X}^G(H) \to Z^\wedge$ by η, we have the following commutative diagram

$$\begin{tikzcd}
& G^\wedge \arrow[dl, "\varkappa"'] \arrow[dr, "\lambda_G"] & \\
\mathfrak{X}^G(H) \arrow[rr, "\eta"] & & Z^\wedge \\
& H^\wedge \arrow[ul, "\pi"] \arrow[ur, "\lambda_H^*"'] &
\end{tikzcd}$$

Now λ_H^* is open, by hypothesis, and π is a continuous surjection (1.2), so η is an open map. But $\varkappa$ is an open map (1.2), so we conclude that λ_G is also an open map, and thus the theorem is proven.

The following corollary answers a question posed to one of the authors by I. Schochetman. We remark that the answer can also be seen from an analysis of the Plancherel measure (see [8, (4.5)]).

(1.5) Corollary: Let G be a $[Z]$-group. Then G is non-compact if and only if for some $\chi \in Z^\wedge$ the fiber $\lambda^{-1}(\chi)$ is nowhere dense. In this case $\lambda^{-1}(\chi)$ is nowhere dense for all $\chi \in Z^\wedge$.

Proof: Since λ is an open continuous map, $\lambda^{-1}(\chi)$ has non-empty interior if and only if $\{\chi\}$ has non-empty interior. This, of course, occurs (for one χ or for all χ) if and only if $Z^\wedge$ is discrete, hence if and only if Z, and therefore G, are compact.

The preceding theory of course depends heavily on the theory of induced representations, as well as the fact, proved in [10], that each irreducible continuous unitary representation of a $[Z]$-group G is finite-dimensional. One can, however, give an independent proof of this fact, using the structure theory for $[Z]$-groups together with well-known general results about unitary representations; in fact, by a similar application of the Mackey procedure one can get an estimate on the degree of the irreducible representations of G. Before dealing with this, however, we give an elementary lemma whose full strength will not be needed until somewhat later.

(1.6) LEMMA: (i) Let G be a $[Z]$-group of the form $G=KZ$, where K is a compact subgroup. Then G is topologically isomorphic to a quotient of the direct product $K\times Z$. Moreover, $G^\wedge$ is (homeomorphic to) an open and closed subset of $K^\wedge \times Z^\wedge$.

(ii) In fact, if M is a $[Z]$-group, and N is a compact normal subgroup, then $(M/N)^\wedge$ is (homeomorphic to) an open and closed subset of $M^\wedge$.

PROOF: (i) To prove the first statement it clearly suffices to show that the continuous epimorphism $\varphi\colon K\times Z \to G$, $\varphi(k,z)=kz$, is open (the map is in fact a homomorphism since $kz=zk$). Thus it suffices to show that the image of any open rectangle $U\times V \subseteq K\times Z$ is open, where we may assume that V has compact closure. For this, consider the compactly generated subgroup Z^* generated by V^-. Z^* is open, so KZ^* is open in G, by the argument of Lemma (1.3); on the other hand the restriction of φ to a map of $K\times Z^*$ onto KZ^* is open, by the open mapping theorem (since $K\times Z^*$ is obviously σ-compact). Thus UV is open in G, and φ is an open map. The last statement of (i) follows from (ii) (let $M=K\times Z$, and $N=\ker\varphi$).

(ii) If $\varrho \in M^\wedge$, then $\varrho\in (M/N)^\wedge$ if and only if $\chi_\varrho \equiv d_\varrho$ on N. Since this condition is preserved under uniform convergence on compacta (in fact, even under pointwise convergence), $(M/N)^\wedge$ is closed in $M^\wedge$. On the other hand, if $\varrho_0\in (M/N)^\wedge$ and $\|\chi_\varrho - \chi_{\varrho_0}\|_N<1$, then $(\chi_\varrho)_N \equiv (\chi_{\varrho_0})_N \equiv d_{\varrho_0}$ (by [8, Theorem 5.1]) so $\varrho\in (M/N)^\wedge$. Thus $(M/N)^\wedge$ is open.

(1.7) PROPOSITION: Let G be a $[Z]$-group, and $G=V\times M$, where M contains a compact open normal subgroup K, as in the structure theorem. If $\varrho\in G^\wedge$, then there is a $\sigma\in K^\wedge$ such that $d_\varrho \mid [G\colon KZ]\, d_\sigma$.

PROOF: Observe first that KZ, as a factor group of $K\times Z$, has only finite-dimensional irreducible representations; in fact, each $\varrho\in (KZ)^\wedge$ is of the form $\varrho=\sigma\otimes\chi$, with $\sigma\in K^\wedge$ and $\chi\in Z^\wedge$, so $d_\varrho = d_\sigma$. Moreover, $[G\colon KZ]$ is finite, so by [18] every $\varrho\in G^\wedge$ is finite-dimensional. We prove the divisibility relation by induction on the length r of the composition series in the abelian group $G/(KZ)$, as in (1.4). In case $r=0$, that is, $G=KZ$, then the above evidently implies that $d_\varrho \mid [G\colon KZ]\, d_\sigma$. Now let H be an open subgroup of G, containing KZ, such that $[G\colon H]=p$, a prime. If $\varrho\in G^\wedge$, then (1.1) shows that either ϱ restricts irreducibly to a $\tau\in H^\wedge$, hence $d_\varrho = d_\tau$, or $\varrho=\tau^G$ for some $\tau\in H^\wedge$, hence $d_\varrho = p d_\tau$. In either case $d_\varrho \mid [G\colon H] d_\tau$. Since by induction we may assume $d_\tau \mid [H\colon KZ]\, d_\sigma$ for some $\sigma \in K^\wedge$, the result follows.

§ 2. A DUALITY THEOREM

In this section we prove a duality theorem, in the form of a dictionary, which describes topological group theoretic properties of the $[Z]$-group G in terms of topological properties of its dual, $G^\wedge$. The correspondences listed below generalize and extend the classical Pontrjagin correspondences for abelian groups (and the proofs depend on the abelian theory). Some results of an analogous nature have been obtained by E. KANIUTH [13] for discrete groups. Recently a duality theory has been formulated for general locally compact groups (see [4], [26], [27], as well as [12]); however, correspondences such as those listed below cannot fall within the scope of this theory.

(2.1) LEMMA: Let G be a locally compact group, and H a closed normal subgroup. Then G is (1) first countable, (2) second countable, or (3) σ-compact if and only if both H and G/H have the same property. Moreover, G satisfies (2) if and only if it satisfies (3) and (1).

PROOF: (1) If G is first countable then, evidently, so are H and G/H. Conversely, let $\{W_i\}$ be a countable neighborhood basis for the identity in G/H, and $\{V_j\}$ a countable family of neighborhoods of 1 in G such that $\{V_j \cap H\}$ is a countable neighborhood basis at the identity in H. Then $\{V_j \cap \pi^{-1}(W_i)\colon i, j = 1, 2, \ldots\}$ is a countable family of neighborhoods of 1 in G (where $\pi\colon G \to G/H$ is the canonical projection). Moreover, $\bigcap_{i,j} V_j \cap \pi^{-1}(W_i) = \{1\}$. Call this countable family $\{U_n\}$. Let U_0 be a fixed compact neighborhood of 1 in G. Replacing, if necessary, U_n by $U_n \cap U_0$, we may assume that $U_n \subset U_0$; by local compactness we may evidently also assume that $\cap\, U_n^- = \{1\}$; finally, by considering finite intersections, we may assume that the U_n's are nested. Now, if U is any neighborhood of 1 in G, $U \subset U_0$, then $U_n^- \subset U$ for some n. For otherwise $U_n^- \cap (U_0 - U)$ is non-empty for each n, so the family $\{U_n^- \cap (U_0 - U)\}$ of closed sets in the compact space U_0 has the finite intersection property. But the complete intersection of these sets is empty, which contradicts the compactness of U_0.

(3) Clearly if G is σ-compact then so are H and G/H. Conversely, suppose H, G/H are σ-compact. If K is a compact set in G/H then by local compactness there is a compact set C in G such that $\pi(C) = K$. Since $\pi^{-1}(K) = CH$, and H is σ-compact, it follows that $\pi^{-1}(K)$ is σ-compact. Therefore if G/H is σ-compact then so is G.

We show next that for any locally compact group G, conditions (3) and (1) together are equivalent to (2). In fact, if G is second countable it is certainly first countable, and by local compactness it is σ-compact. Conversely, if G is first countable then it is metrizable; if in addition G is the countable union of compact metrizable subspaces, then each of these subspaces, hence also G, has a countable dense subset. As a separable metric space G is second countable. This proves the last statement. Moreover, in view of the above, this shows that G is second countable if and only if both H and G/H are second countable.

(2.2) LEMMA: Let G be a $[Z]$-group, and $\varrho \in G^\wedge$. Then $\mathfrak{C}_\varrho$, the connected component of ϱ in $G^\wedge$, is the equivalence class defined by the equivalence relation $(\chi_\sigma)_P \equiv (\chi_\varrho)_P$, where P is the periodic subgroup of G.

In particular, $\mathfrak{C}_1$, the connected component of the trivial representation, equals $(G/P)^\wedge$.

Proof: For any closed normal subgroup N of G, let $\mathfrak{D}_{\varrho,N}=\{\sigma\in G^\wedge\colon (\chi_\sigma)_N\equiv(\chi_\varrho)_N\}$; then we must show that $\mathfrak{D}_{\varrho,P}=\mathfrak{C}_\varrho$. Now if G/N is abelian, then it follows from a result of Kaniuth and Schlichting [15] that $\mathfrak{D}_{\varrho,N}=\{\alpha\otimes\varrho\colon\alpha\in(G/N)^\wedge\}$. In particular, $\mathfrak{D}_{\varrho,P}$ is the image of $(G/P)^\wedge$ under the continuous map $\alpha\to\alpha\otimes\varrho$ of $(G/P)^\wedge\to G^\wedge$ (since G/P is abelian [9, Theorem 5.4]). But G/P is an aperiodic abelian group, so by abelian group theory [21, p. 277] $(G/P)^\wedge$ is connected, and thus $\mathfrak{D}_{\varrho,P}$ is connected. Therefore $\mathfrak{D}_{\varrho,P}\subseteq\mathfrak{C}_\varrho$. To establish the reverse inclusion, suppose K is any compact normal subgroup of G. Then $\mathfrak{D}_{\varrho,K}$ is clearly a closed subset of $G^\wedge$, and it is also open; in fact, if $\sigma\in G^\wedge$ and $\|\chi_\sigma-\chi_\varrho\|_K<1$, then $(\chi_\sigma)_K\equiv(\chi_\varrho)_K$ by [8, Theorem 5.1]. Therefore $\mathfrak{C}_\varrho\subseteq\mathfrak{D}_{\varrho,K}$ for each compact normal subgroup K of G. Since P is the union of such subgroups K, $\mathfrak{C}_\varrho\subseteq\mathfrak{D}_{\varrho,P}$. This completes the proof.

Remark: It follows from the result in [15] together with (2.2) that, in fact, the components in $G^\wedge$ can also be described by the equivalence relation $(d_\sigma^{-1}\chi_\sigma)_P=(d_\varrho^{-1}\chi_\varrho)_P$. Also we note explicitly that the component $\mathfrak{C}_\varrho$ is thus the orbit of ϱ under the action of $(G/P)^\wedge$ on $G^\wedge$ given by $(\alpha,\varrho)\longmapsto\alpha\otimes\varrho$.

We are now ready to state the main theorem of this section. For convenience we make a preliminary definition, following Pontrjagin [21]. If G is a locally compact group, we say G has property (L), or is an (L)-group, if each compact subset C of G is contained in a compactly generated, open, normal subgroup H such that G/H is torsion free.

(2.3) Theorem: Let G be a $[Z]$-group. Then G is

(1) first countable
(2) second countable
(3) σ-compact
(4) compactly generated
(5) compact
(6) discrete
(7) periodic
(8) aperiodic
(9) has finite periodic subgroup
(10) an (L)-group

if and only if the dual space $G^\wedge$ is

(1′) σ-compact
(2′) separable metrizable
(3′) first countable
(4′) locally Euclidean
(5′) discrete
(6′) compact
(7′) totally disconnected
(8′) connected
(9′) has finitely many connected components
(10′) locally connected.

PROOF: (1) If $G^\wedge$ is σ-compact then so is $Z^\wedge$, since $\lambda: G^\wedge \to Z^\wedge$ is continuous and surjective [10]. It follows from the abelian theory [22, ch. 4, § 2.3] that Z is then first countable. Now $\lambda^{-1}(1) = (G/Z)^\wedge$ is discrete, and $G^\wedge$ is σ-compact, so $\lambda^{-1}(1)$ is countable. By the Peter-Weyl Theorem it follows that $L_2(G/Z)$ is separable, so G/Z is second countable, and in particular first countable. It then follows from (2.1) that G is first countable.

Conversely, suppose G is first countable. If G is of the form $K \times Z$, where K is compact, then K and Z are first countable. Hence by abelian theory $Z^\wedge$ is σ-compact [22, ch. 4, § 2.3]; also K is then compact metrizable, hence separable, so $K^\wedge$ is countable. It follows that $G^\wedge = K^\wedge \times Z^\wedge$ is σ-compact, and thus the result holds if $G = K \times Z$. If G is of the form KZ, no longer necessarily a direct product, then G is a quotient group of $K \times Z$ (lemma (1.6)) and therefore $G^\wedge$ is a closed subset of $K^\wedge \times Z^\wedge$. Thus in this case also, first countability of G implies σ-compactness for $G^\wedge$. As in the proof of (1.4), we complete the present proof by induction on the length r of the normal series $G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_r = KZ$, where K is the compact subgroup given by the structure theorem. We have just dealt with the case $r = 0$, that is, $G = KZ$; we may therefore assume that we are dealing with the inductive step, that is, G contains a normal subgroup H of prime index, such that the assertion holds for H. Thus $H^\wedge$ is σ-compact (since H is first countable). Since $\pi: H^\wedge \to \mathfrak{X}^G(H)$ is a continuous surjection (1.2) it follows that $\mathfrak{X}^G(H)$ is σ-compact. Finally, since $\varkappa$: $G^\wedge \to \mathfrak{X}^G(H)$ is a proper mapping (1.2), it follows that $G^\wedge$ is also σ-compact.

(3) Assume $G^\wedge$ is first countable. Since $Z^\wedge$ is locally homeomorphic to $G^\wedge$ (1.4), $Z^\wedge$ is first countable, and therefore Z is σ-compact. Since by definition G/Z is compact, it follows from (2.1) that G is σ-compact. The converse follows by reversing the argument.

(2) If $G^\wedge$ is separable and metrizable then it is first countable and σ-compact hence by (1) and (3) G is σ-compact and first countable. It follows from (2.1) that G is second countable. Conversely, suppose G is second countable. An argument virtually identical to that in the proof of (1) shows that it is enough to deal with the inductive step. Thus we may assume G contains a normal subgroup H of prime index, and that $H^\wedge$ is separable and metrizable. Now a second countable locally compact Hausdorff space is metrizable [2a, ch. 9, § 2.9, Cor. of Prop. 16] and separable, so it suffices to prove that $G^\wedge$ is second countable. Now $\pi: H^\wedge \to \mathfrak{X}^G(H)$ is a continuous open surjective mapping, so the image of a countable basis for the topology of $H^\wedge$ is a countable basis for the topology of $\mathfrak{X}^G(H)$. Thus $\mathfrak{X}^G(H)$ is second countable, hence so are $\mathfrak{X}^G(H)_I$ and $\mathfrak{X}^G(H)_{II}$ (see (1.2)). Since $\varkappa_{II}$: $(G^\wedge)_{II} \to \mathfrak{X}^G(H)_{II}$ is a homeomorphism, it follows that the open subset $(G^\wedge)_{II}$ of $G^\wedge$ is second countable. Moreover, $\varkappa_* : (G^\wedge)_I \to (H^\wedge)_I$ is a covering projection with finite fiber. If V is any evenly covered open neighborhood in $(H^\wedge)_I$, then $\varkappa_*^{-1}(V)$ is an open second countable subspace of $(G^\wedge)_I$ (being the finite union of open sets homeomorphic with V). Since by the Lindelöf property $(H^\wedge)_I$ can be written

as the countable union of evenly covered open neighborhoods, it follows that $(G^\wedge)_I$ is the countable union of open second countable subspaces, and therefore is itself second countable. Finally, since $(G^\wedge)_I$ and $(G^\wedge)_{II}$ are open, $G^\wedge$ is second countable.

(4) If G is compactly generated then Z is also compactly generated, since G/Z is compact (see [2b, ch. 7, § 3.2, Lemma 3]). By abelian theory [20, Corollary 1 of Theorem 2.5] it follows that $Z^\wedge$ is locally Euclidean. Since $\lambda: G^\wedge \to Z^\wedge$ is a local homeomorphism, it follows that $G^\wedge$ is locally Euclidean. The converse follows by reversing the argument, and using the fact that $G = UZ$, as in [10].

(5) [1]) If G is compact it is well-known, and follows from [8, Theorem 5.1] that $G^\wedge$ is discrete. Conversely, if $G^\wedge$ is discrete, or if $G^\wedge$ has even one isolated point, then G is compact (for example, by (1.5)).

(6) If $G^\wedge$ is compact then so is the maximal ideal space $\mathfrak{M}(\mathfrak{Z}(L_1))$ of the center of $L_1(G)$ [8, Theorem (6.6)], and therefore $\mathfrak{Z}(L_1)$ has an identity f [23, (3.6.6)]. We show that f is an identity for $L_1(G)$. In fact, if $h \mapsto h^\sim$ is the Gelfand transform on $\mathfrak{Z}(L_1)$ (after identifying $G^\wedge$ with the maximal ideal space of $\mathfrak{Z}(L_1)$) then $T_h(\varrho) = h^\sim(\varrho)I$ for $\varrho \in G^\wedge$, $h \in \mathfrak{Z}(L_1)$ [8, section 6]. Therefore $T_f(\varrho) = I$ for all $\varrho \in G^\wedge$, so if $g \in L_1(G)$ is arbitrary, then $T_{f\star g}(\varrho) = T_f(\varrho)T_g(\varrho) = T_g(\varrho)$. By the Gelfand-Raikov theorem, $f \star g = g$. Since $L^1(G)$ has an identity, G is discrete. The converse is clear.

(7) If G is periodic, then by (2.2) each component of $G^\wedge$ is a point, so $G^\wedge$ is totally disconnected. Conversely, if $G^\wedge$ is totally disconnected, then $\mathfrak{C}_1 = \{1\} = (G/P)^\wedge$, so $G = P$.

(8) If G is aperiodic, then $\mathfrak{C}_1 = G^\wedge$ by the remark following (2.2), so $G^\wedge$ is connected. Conversely, if $G^\wedge$ is connected, then $G^\wedge = (G/P)^\wedge$, so each irreducible representation of G is trivial on P. By the Gelfand-Raikov theorem, $P = (1)$. We notice, incidentally, that (8) (or (8′)) implies G abelian.

(9) If P is finite, then PZ has representations of bounded degree, hence so does G [18], since $[G: PZ]$ is finite. If $\varrho \in G^\wedge$, then ϱ_P is determined by the (orbit of) irreducibles it contains and by its degree. Since both these parameters range only over finite sets, it follows from (2.2) that $G^\wedge$ has only finitely many components. Conversely, if $G^\wedge$ has only finitely many components, then each is open. Therefore $(G/P)^\wedge$ is open in $G^\wedge$, so since $P \supset (G')^-$, $(G/(G')^-/P/(G')^-)^\wedge$ is open in $(G/(G')^-)^\wedge$. Thus by abelian theory $P/(G')^-$ is compact, hence P is compact. Now each irreducible of P is a component of the restriction to P of some irreducible of G [10, Theorem 5.1]. Since there are only finitely many such inequivalent restrictions, and each is finite-dimensional, $P^\wedge$ is finite and therefore so is P.

Remark: Since the connected component $\mathfrak{C}_1$ of 1 in $G^\wedge$ equals $(G/P)^\wedge$, for G a $[Z]$-group, we actually have a somewhat stronger statement than

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1) This result has been generalized to $[FIA]_B^-$ groups by E. Kaniuth [14] and R. Mosak [19], and to more general locally compact groups by L. Baggett [1] and J. Liukkonen [16].

(7). In fact, the following are equivalent: G is periodic; $G^\wedge$ is totally disconnected; $\mathfrak{C}_1$ is trivial. In view of the above, to prove this we need only show that $\mathfrak{C}_1$ trivial implies $G=P$. But $\mathfrak{C}_1$ trivial implies $(G/P)^\wedge$ is trivial, and by the Gelfand-Raikov theorem this implies $G=P$.

(10) Suppose first that G has property (L), and let F be a compact set in Z. Let U be a compact neighborhood of the identity in G, such that $G=UZ$, as in [10]. Then UF is a compact set in G, so by hypothesis there exists a compactly generated open normal subgroup H of G such that $H \supseteq UF$, and such that G/H is torsion free. Then $H \cap Z$ is clearly an open subgroup of Z containing F. Moreover, $H/H \cap Z \cong HZ/Z \subseteq G/Z$; since HZ is open and therefore closed in G, and G is a $[Z]$-group, it follows that $H/H \cap Z$ is compact. Since H is compactly generated so is $H \cap Z$ [2b, ch. 7, § 3.2, Lemma 3]. In addition, $Z/H \cap Z \cong ZH/H \subseteq G/H$, so $Z/H \cap Z$ is torsion-free. Therefore Z has property (L). By a result of Fan [5] $Z^\wedge$ is locally connected. Since $\lambda: G^\wedge \to Z^\wedge$ is a local homeomorphism it follows that $G^\wedge$ is locally connected.

Conversely, suppose $G^\wedge$ is locally connected. Since $(G/(G')^-)^\wedge$ is open and closed in $G^\wedge$ (see (1.6)), $(G/(G')^-)^\wedge$ is also locally connected. It follows from Fan's result [5] that $G/(G')^-$ has property (L). Let C be a compact set in G, and let $\alpha: G \to G/(G')^-$ be the canonical epimorphism. Then the compact set $\alpha(C)$ is contained in an open compactly generated subgroup $H/(G')^-$ of $G/(G')^-$ with the property that the corresponding quotient group is torsion free. Since $H=\alpha^{-1}(H/(G')^-)$, and $(G')^-$ is compact, H is an open compactly generated subgroup of G. H evidently contains $\alpha^{-1}(\alpha(C)) \supseteq C$, and $H \supseteq (G')^-$, so H is normal. Finally, $G/H \cong G/(G')^-/H/(G')^-$, and is therefore torsion-free. Thus G has property (L).

Remark: Since $G^\wedge$ is locally connected if and only if $Z^\wedge$ is locally connected, the above result shows that for a $[Z]$-group G, G has property (L) iff Z has property (L). An independent proof of the sufficiency does not seem at all obvious.

(2.4) Corollary: Let G be a $[Z]$-group. If G is discrete and finitely generated, then $G^\wedge$ has a finite number of components.

Proof: If G is finitely generated or even compactly generated, then $P(G)$ is compact [9, Corollary 1 to Theorem 5.5], hence finite if G is discrete. The corollary then follows from (9) in the theorem.

(To be continued)